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Does Thermodynamics Rule Out the Existence of Cosmological Singularities?

Marcelo Schiffer[1]

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Recently, Bekenstein showed that a singularity in the FRW radiation-dominated cosmological model is inconsistent with the "entropy bound," a new thermodynamic law he put forward a few years ago. In this paper we generalize his results and show that, regardless of model peculiarities, the existence of cosmological singularities is incompatible with thermodynamics.

1. INTRODUCTION

The advent of general relativity marks the beginning of modern cosmology. The study of homogeneous and isotropic models allowed science to make remarkable predictions concerning the universe we live in, such as Hubble's law, the helium abundance, and the cosmic background thermal radiation; and to describe early stages of evolution of the universe. Unfortunately, these models also predict that, as we go back in time, matter energy density and pressure increase indefinitely, until a point where the curvature blows up is reached—a spacetime singularity. At this point all physical laws should break down.

Of course, one may wonder whether such a behavior is a particularity of the models studied, or if it is rather a general disease that afflicts general relativity. With such worries in mind, many researchers scrutinized non-isotropic (Kasner, 1921; Misner, 1969; Khalatnikov and Lifshitz, 1963; Belinsky *et al.*, 1970) looking to circumvent the cosmological singularity.

Soon it became clear that all efforts toward nonsingular cosmological solutions are condemned to fail. Through a set of powerful theorems, Penrose and Hawking (Hawking and Ellis, 1980) proved the remarkable result that, if matter satisfies very general conditions like the "strong energy condition" (essentially, that pressure and energy density are positive) and if global causality holds, a singularity in spacetime must follow.

[1]Instituto de Fisica Teórica, 01405 São Paulo, SP-Brazil.

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Obviously, the cosmological singularity may be avoided through some violation of the strong energy condition. This may be accomplished by introducing negative energy or pressure in some models. This is the way inflationary cosmology works (Guth, 1981). However, this resolution relies on a very particular model and does not answer the fundamental question: Do the known laws of physics rule out the existence of spacetime singularities? The widespread belief is that, when a self-consistent theory of quantum gravity is available, all these matters will be settled.

Here we pose the question of whether thermodynamic reasoning has some predictive power concerning the existence of the cosmological singularities allowing us to circumvent the present lack of a microscopic theory of gravity. This situation is very akin to the conceptual and experimental problems science faced at the end of the previous century concerning the blackbody radiation. We recall that the lack of a microscopic description of radiation was not an impassable barrier for important predictions—witness Wien's displacement law and the Stefan–Boltzmann law, which were obtained solely based on thermodynamic considerations. Before delving into this question, let us do a thermodynamic detour.

The second law of thermodynamics affirms that the entropy of a closed system tends to a maximum, without stating how large this could be. On the other hand, a common intuitive feeling is that this entropy should be limited in terms of the system's size and energy, which is suggested by the limited phase space available for such a system. The formal translation of this intuitive feeling arose some 10 years ago as Bekenstein (1981*a*) worked out a consistency condition between black hole thermodynamics and ordinary statistical physics. He concluded that the condition for the generalized second law (ordinary matter + B.H.) to hold always is that the entropy of *any* system with largest linear dimension R and proper energy E must be limited by

$$S/E \leq 2\pi R/\hbar c \tag{1}$$

The existence of this bound has been corroborated in the realm of quantum field theory without interactions (Bekenstein, 1981*a*,*b*, 1983; Schiffer and Bekenstein, 1989), with interactions (Bekenstein and Guendelman, 1987; Schiffer, 1988), of string theory (Bowick *et al.*, 1986), nuclear physics (Schiffer *et al.*, 1990), and also in the presence of strong gravitational fields (Sorkin *et al.*, 1981). Moreover, the entropy bound predicts a limit on the rate wherein information may be conveyed between an emitter and a receiver, and limits the number of families of elementary particles. These results are either in agreement with experiment or have been derived by other means (Bekenstein, 1982; Bekenstein and Schiffer, 1990). Equation (1) is to be regarded as a physical law which supplements the second law.

Answering the above question, recently, Bekenstein (1989) applied the entropy bound for an observer living in a radiation-dominated Friedman–Robertson–Walker model. As discussed by him, due to the fact that any observer is causally disconnected from any region beyond his particle horizon (P.H.), *the cosmological counterpart for a closed system should be the spacetime region within the observer's particle horizon.* He showed that for this "system" the entropy bound is violated as the observer approaches the initial singularity. Thus, thermodynamic reasoning has the predictive power of ruling out the existence of a singularity in the radiation-dominated FRW model.

The purpose of this paper is to push forward Bekenstein's idea and to prove a general, model-independent result showing the inconsistency between the existence of a cosmological singularity and thermodynamics. For this end, following Belinsky *et al.* (1970), we expand the metric around a spacetime singularity and apply the entropy bound to the particle horizon interior defined by this asymptotic expansion. Moreover, we *do not* specify the universe matter content, but solely impose very general restrictions upon it, namely we (i) demand pressure and energy density positiveness, (ii) forbid causality violation, and (iii) assume the matter entropy density to be always bounded from above by that of the corresponding thermal radiation. Then we shall conclude that, *for any reasonable universe (in the sense that matter satisfies the conditions listed above) simultaneous consideration of general relativity and the entropy bound forbids the existence of spacetime singularities!*

This paper is organized as follows. In Section 2 we parallel the Belinsky *et al.* (1970) expansion of the metric around a spacetime singularity and constrain the range of the parameters which govern this metric, imposing the above-mentioned conditions on the matter content and assuming the three-curvature tensor to be finite. In Section 3, we apply the entropy bound to the region inside the observer's particle horizon. In Section 4, we study a toy model of matter falling toward the Schwarzschild singularity, showing explicitly violation of the entropy bound in the context of black-hole singularities. Next, in Section 5, we analyze under which conditions the three-curvature may be neglected, and amend our results for circumstances where this condition is not fulfilled. In Section 6, we discuss the prospects of the present research.

2. THE METRIC NEAR THE SINGULARITY

It is widely accepted that near *any* timelike singularity the line element in a synchronous reference system has Belinsky *et al.* (1970) asymptotic form

$$ds^2 = -dt^2 + h_{ij}(x, t)\, dx_j\, dx_j \tag{2}$$

with

$$h_{ij} = a^2 l_i l_j + b^2 m_j m_j + c^2 n_j n_j \tag{3}$$

Here **l**, **m**, and **n** are a dreibein basis, which are functions of the space coordinates (for the purpose of vectorial operations **l**, **m**, and **n** should be regarded as vectors in Cartesian coordinates). The group of motions in this spacetime satisfy a Lie algebra whose structure constants may be classified in terms of three parameters λ, μ, and ν:

$$\lambda = \mathbf{l}\cdot\nabla\times\mathbf{l}, \qquad \mu = \mathbf{m}\cdot\nabla\times\mathbf{m}, \qquad \nu = \mathbf{n}\cdot\nabla\times\mathbf{n} \tag{4}$$

with $\mathbf{l}\cdot(\mathbf{m}\times\mathbf{n}) = 1$.

As these observers approach the singularity, they get closer to each other with an expansion rate

$$\theta = v^{\mu}{}_{;\mu} = \frac{\partial v^{\mu}}{\partial x^{\mu}} + v^{\mu}\frac{\partial \ln\sqrt{h}}{\partial x^{\mu}} = \frac{\partial \ln\sqrt{h}}{\partial t} \tag{5}$$

The second fundamental form is defined as

$$\chi_{ij} = \frac{1}{2}\frac{\partial h_{ij}}{\partial t} \tag{6}$$

whose trace is

$$\chi \equiv \chi^i_i = \tfrac{1}{2}h^{ij}\frac{\partial h_{ij}}{\partial t} = \frac{\partial \ln\sqrt{h}}{\partial t} \tag{7}$$

having the straightforward physical interpretation of being the expansion rate (5).

Einstein's equations may be cast in the form (Landau and Lifshitz, 1970)

$$R^0_0 = -\frac{\partial}{\partial t}\chi - \chi^i_j\chi^j_i = 8\pi(T^0_0 - \tfrac{1}{2}T) \tag{8}$$

$$R^0_i = (\chi^j_{i;j} - \chi^j_{j;i}) = 8\pi T^0_i \tag{9}$$

$$R^i_j = -{}^3R^i_j - \frac{1}{\sqrt{h}}\frac{\partial}{\partial t}\sqrt{h}\,\chi^i_j = 8\pi(T^i_j - \tfrac{1}{2}\delta^i_j T) \tag{10}$$

where ${}^3R^0_i$ stands for the three-geometry Ricci tensor. If rightmost part of (8) is semi-positive definite (strong energy condition), it is rather trivial to show the inequality (Landau and Lifshitz, 1970)

$$\frac{\partial}{\partial t}\chi^{-1} \geq \frac{1}{3} \tag{11}$$

Assuming χ to be initially positive at some t_0, as time decreases, so must χ^{-1}, until it vanishes. At this point, θ diverges corresponding to the formation of a caustic at some earlier time. This point need not necessarily correspond to a physical singularity, and may be fictitious, owing its existence to the bad choice of coordinate system. Here we shall assume this point to be singular, and choose the origin of time to coincide with it. This fact, in combination with equation (8), implies that h must vanish as one approaches the singularity with a power of t that never exceeds 6 (Landau and Lifshitz, 1970). Therefore,

$$h \approx t^{2n} \tag{12}$$

$$\chi = \frac{1}{2}\frac{\partial \ln h}{\partial t} = \frac{n}{t}; \qquad 0 < n \leq 3 \tag{13}$$

It is convenient to express $a = t^{\alpha_1}$, $b = t^{\alpha_2}$, and $c = t^{\alpha_3}$, where α_i are arbitrary functions of the coordinates. The corresponding volume element is

$$h = t^{2(\alpha_1+\alpha_2+\alpha_3)} \tag{14}$$

Comparison of (12) and (14) tells us that

$$\alpha_1(x) + \alpha_2(x) + \alpha_3(x) = n \tag{15}$$

For the purpose of solving Einstein's equations (8)-(10) we evaluate the second fundamental form

$$\chi^i_j = \frac{\dot{a}}{a} l^i l_j + \frac{\dot{b}}{b} m^i m_j + \frac{\dot{c}}{c} n^i n_j \tag{16}$$

In the present section we shall assume that, as the singularity is approached, matter pressure and energy density diverge while the three-dimensional Ricci tensor remains finite. Thus, near the singularity, equation (10) reads

$$\frac{\partial}{\partial t}\chi^i_j + \chi\chi^i_j = -8\pi(T^i_j - \tfrac{1}{2}\delta^i_j T) \tag{17}$$

where the definition of χ [equation (7)] was used.

In the particular coordinate system where the spatial part of the energy-momentum tensor is diagonal we have

$$T^i_j = -p_i \delta^i_j$$
$$T^0_0 = \rho \tag{18}$$

where p_i and ρ stand for the principal pressures and energy density as measured by this observer. In light of (18) the trace of (17) reads

$$\frac{\partial}{\partial t}\chi + \chi^2 = 12\pi(\rho - \bar{p}) \tag{19}$$

where $\bar{p}$ stands for the mean pressure $\bar{p} = (p_1 + p_2 + p_3)/3$. Having in mind

(13), we obtain for this equation

$$\rho - \bar{p} = \frac{n(n-1)}{12\pi t^2} \tag{20}$$

On causality grounds, we must have $\rho \geq \bar{p}$, constraining n to be in the interval

$$1 < n \leq 3 \tag{21}$$

(henceforth the $n = 1$ case will be excluded from our considerations, since it either corresponds to vacuum solutions or to completely stiff matter). Next, after inserting (13), (16), and (18), we find for (8)

$$(n/t^2) - [(\dot{a}/a)^2 + (\dot{b}/b)^2 + (\dot{c}/c)^2] = 4\pi(\rho + 3\bar{p}) \tag{22}$$

The term in brackets may be reexpressed in terms of the α_i,

$$(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)t^{-2} + (\dot{\alpha}_1^2 + \dot{\alpha}_2^2 + \dot{\alpha}_3^2)(\ln t)^2 + 2(\alpha_1\dot{\alpha}_1 + \alpha_2\dot{\alpha}_2 + \alpha_3\dot{\alpha}_3)\frac{\ln t}{t}$$

If the three-curvature is finite at the origin, then, according to (20), the right-hand side of (22) behaves exactly as t^{-2}. Thus, close to the singularity, the α's are asymptotically time independent. Defining $\boldsymbol{\alpha}(\mathbf{x}) \equiv (\alpha_1, \alpha_2, \alpha_3)$, we find for equation (22)

$$(\rho + 3\bar{p}) = \frac{n - \boldsymbol{\alpha}^2}{4\pi t^2} \tag{23}$$

We may solve (20) and (23) for the $\bar{p}$ and ρ, obtaining

$$\rho = \frac{n^2 - \boldsymbol{\alpha}^2}{16\pi t^2} \tag{24}$$

and

$$\bar{p} = \frac{n(4-n) - 3\boldsymbol{\alpha}^2}{48\pi t^2} \tag{25}$$

Positiveness of local pressure and energy density constrains α,

$$\boldsymbol{\alpha}^2 \leq \frac{n(4-n)}{3} \leq n^2 \tag{26}$$

As shown in Figures 1-3, the α's lie on the circumference [the circumference of solutions (C.S.)] given by the intersection of the plane (15) with the sphere of radius $|\boldsymbol{\alpha}|$ [represented by the dashed lines in the figures. The dashed triangle represents the intersection of this plane with the planes $\alpha_i = 0$).] Obviously, the radius $|\boldsymbol{\alpha}|$ must be larger than the distance of the plane to the center of the sphere, i.e., larger than $n/\sqrt{3}$,

$$\frac{n^2}{3} \leq \boldsymbol{\alpha}^2 \leq \frac{n(4-n)}{3} \tag{27}$$

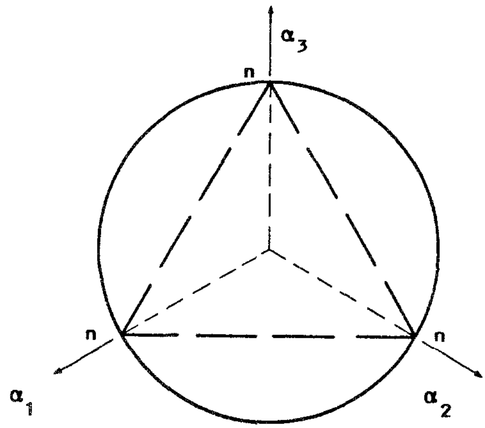

Fig. 1

demanding $n \leq 2$. Therefore, for any solution

$$1 < n \leq 2 \tag{28}$$

Useful information may be obtained by inspecting these figures, allowing the classification of three distinct situations:

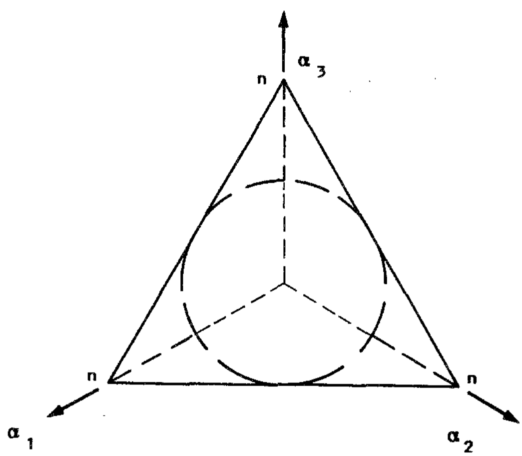

Fig. 2

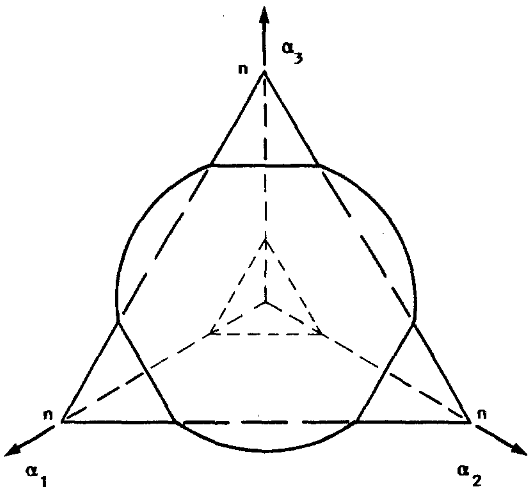

Fig. 3

(i) This circumference circumscribes this triangle. In this case, either two components of α are positive and the third is negative, or two vanish and the third is one. This situation occurs only when $n=1$, which has been excluded from the very beginning (see Figure 1).

(ii) This circumference lies inside the triangle and all α's are positive. In this situation $n^2/3 \leq \boldsymbol{\alpha}^2 \leq n^2/2$ (see Figure 2) and the universe expands in all directions.

(iii) The circumference cuts the triangle at six points. The part of the circle inside the triangle corresponds to case (ii), while for the rest of the circle we have two positive and one negative α, so that the universe expands in two directions and contracts in the third. In this situation $n^2/2 \leq \boldsymbol{\alpha}^2 \leq n(4-n)/3$ (see Figure 3).

At this point, some remarks are in order. The value of n is fixed through the equation of state $\bar{p}=\bar{p}(\rho)$ and the relation between the components of the "vector" $\boldsymbol{\alpha}$. Thus, for instance, for a traceless energy-momentum tensor, $\boldsymbol{\alpha}^2=n(2-n)$. Further specifying the model to be isotropic, we obtain $n=3/2$, corresponding to a radiation-dominated Friedmanian behavior. Vacuum solutions demand $n=1$ and $\boldsymbol{\alpha}^2=1$, corresponding to Kasner behavior.

A second and important remark: As will be transparent later, we shall be concerned with very small time scales, of the order of Planck's time. Thus, due to the fact that the matter typical relaxation time is orders of magnitude larger than this, matter will only start probing spacetime inhomogeneities much later. This means that sufficiently close to the singularity we should have $T_i^0 \approx 0$. Thus, recalling that the trace of the second fundamental form does not depend on spatial coordinates [see equation (13)], sufficiently close to the singularity [see equation (9)]

$$\chi^j_{i;j} \approx 0 \tag{29}$$

3. THE ENTROPY BOUND AND THE COMOVING OBSERVER

The entropy bound (EB) refers to a closed system whose energy, entropy, and largest linear dimension meanings are, in most applications, straightforward. However, when dealing with a system like the universe, a critical application of the EB is more subtle and demands some wariness. Following Bekenstein's paper, the EB is to be applied to the region limited by the obsever's particle horizon, the one the observer is causally connected to. Thus, the "system's linear dimension" is to be understood as the proper radius R of the 3-sphere that encloses the particle horizon, while the corresponding energy E and entropy S are to be computed on a spacelike hypersurface within the PH. Since the above three parameters are functions of t_0, the observer's comoving time, we state the EB in the form

$$f(t_0) \equiv \frac{S(t_0)\hbar c}{2\pi R(t_0)E(t_0)} \leq 1 \tag{30}$$

In order to find out where the particle horizon lies, it is convenient to move to the principal direction coordinate system (y^i), the one where the metric (3) is locally diagonal

$$\operatorname{diag} h'_{ij} = (a^2, b^2, c^2) \tag{31}$$

In this coordinate system, (29) is equivalent to

$$\frac{\partial \alpha_1}{\partial y^1} = \frac{\partial \alpha_2}{\partial y^2} = \frac{\partial \alpha_3}{\partial y^3} \approx 0 \tag{32}$$

The light cone hypersurface is given by $ds^2 = 0$ or, equivalently, by

$$t^{2\alpha_1}\left(\frac{dy^1}{dt}\right)^2 + t^{2\alpha_2}\left(\frac{dy^2}{dt}\right)^2 + t^{2\alpha_3}\left(\frac{dy^3}{dt}\right)^2 = 1 \tag{33}$$

The solution is a null hypersurface with parametric equation

$$\begin{aligned} y_h^1(t, \mathbf{x}) &= \frac{t_0^{(1-\alpha_1)} - t^{(1-\alpha_1)}}{1-\alpha_1} \cos\theta \\ y_h^2(t, \mathbf{x}) &= \frac{t_0^{(1-\alpha_2)} - t^{(1-\alpha_2)}}{1-\alpha_2} \sin\theta \cos\phi \\ y_h^3(t, \mathbf{x}) &= \frac{t_0^{(1-\alpha_3)} - t^{(1-\alpha_3)}}{1-\alpha_3} \sin\theta \sin\phi \end{aligned} \tag{34}$$

The particle horizon is given by the intersection of (34) with the spacelike hypersurface $t=0$. The corresponding circumscribing 3-sphere proper radius at time t_0 is

$$R=\max(R_h^i)$$
$$R_h^i=\int_0^{y_h^i} ds = t^{\alpha_i}\int_0^{y_h^i} dy^i=\frac{t_0}{1-\alpha_i} \tag{35}$$

Due to the facts that (35) is performed in the direction y^i ($y_\perp^i=0$) and that α^i does not depend on y^i [see equation (32)], the α's are to be evaluated at $y=0$ (henceforth the α's are to be understood as constants).

As already discussed, close to the singularity we should have $T^{0i}\approx 0$, so that the entropy current s^μ is nearly conserved (Misner *et al.*, 1973) and, consequently, an approximate total entropy may be defined

$$S(t)\approx\int_\Sigma s^\mu\, d\Sigma_\mu \tag{36}$$

Had we known some microscopic information concerning the universe matter content, the calculation of the entropy would be straightforward. However, since we are only concerned with the peak value of S/E, the lack of such information can be overcome by resorting to the expedient of rephrasing the EB in terms of the entropy $\tilde{S}$ matter *would have if it were composed of thermal radiation.* This is because, within a given energy budget, matter entropy is bounded from above by that of massless quanta (rest mass reduces the available phase space) in a thermal state. Accordingly,

$$\tilde{s}^0=\tfrac{4}{3}\alpha^{1/4}\rho^{3/4} \tag{37}$$

where is the Stefan–Boltzmann constant. The corresponding energy is

$$E(t)=\int_\Sigma T_0^\mu\, d\Sigma_\mu \tag{38}$$

In order to perform these integrations, we must specify the hypersurface Σ. For the purpose of evaluating $\tilde{S}$, the surface considered is irrelevant, since equation (36) is invariant under deformations of Σ. The same is not true for the energy $E(t)$. It decreases as this surface is deformed toward the future (Bekenstein, 1989). Since we are concerned with the point where the ratio S/E peaks, we shall deform Σ toward the future until it coincides with the null hypersurface (34). By virtue of (15), the volume element reads

$$d\Sigma_0=t^n\, dy^1\, dy^2\, dy^3 \tag{39}$$

Switching to radial coordinates t, θ, ϕ and defining a new time parameter $\zeta = t/t_0$, we perform the angular integrations, and integrate by parts in ζ, obtaining

$$\frac{\tilde{S}}{E} = \frac{8}{3}\left(\frac{\sigma\pi}{n^2 - \alpha^2}\right)^{1/4} t_0^{1/2} \frac{\int_1^{\zeta'} \zeta^{n-3/2}\, dI(\zeta)}{\int_1^{\zeta'} \zeta^{n-2}\, dI(\zeta)}$$

with

$$I(\zeta) = (1 - \zeta^{(1-\alpha_1)})(1 - \zeta^{(1-\alpha_2)})(1 - \zeta^{(1-\alpha_3)}) \tag{40}$$

where $\zeta' = t'/t_0$ is a cutoff time parameter corresponding to the deepest distance the observer may see in the sky ($\zeta' = 0$ corresponds to objects which are close to the particle horizon and are very faint) (Bekenstein, 1989).

Since we are concerned here with the peak of S/E, one should evaluate this function at its maximum, which occurs as $\zeta' \to 1$, namely

$$\lim_{\zeta' \to 1} \frac{\int_1^{\zeta'} \zeta^{n-3/2}\, dI(\zeta)}{\int_1^{\zeta'} \zeta^{n-2}\, dI(\zeta)} = 1$$

Therefore

$$\frac{\tilde{S}}{E} = \frac{8}{3}\left(\frac{\sigma\pi}{n^2 - \alpha^2}\right)^{1/4} t_0^{1/2} \tag{41}$$

This is very much the same result we would have obtained had we divided the entropy and the energy densities. Putting this result together with the calculation of the particle horizon (35) allows us to evaluate $f(t_0)$,

$$f(t_0) = \frac{4}{3\pi}(1 - \alpha_i) N^{1/4} \left(\frac{\sigma\pi}{n^2 - \alpha^2}\right)^{1/4} \left(\frac{t_p}{t_0}\right)^{1/2} \tag{42}$$

Here a slight generalization of (30) was considered in order to accommodate N different species of particles, $t_p = (hG/c^5)^{1/2}$ is Planck's time, and it is understood that the Stefan-Boltzmann constant is expressed in natural units $c = \hbar = 1$, i.e., $\sigma = \pi^2/60$.

Due to the conditions we imposed on the universe matter content, $1/(n^2 - \alpha^2) \geq 3/2n^2 \geq \frac{3}{8}$ [see equations (27) and (28)]. As will be shown in Section 4, the present approach is only valid as long as $\alpha_i \leq (3n - 1)/6$. Thus,

$$f(t_0) > \frac{1}{18}\left(\frac{8N}{5\pi}\right)^{1/4} \left(\frac{t_p}{t_0}\right)^{1/2} \tag{43}$$

For N in the range of 10^2–10^4 species of elementary particles, the numerical factor is of order one, showing that the EB is violated whenever the observer approaches the singularity on a time scale of the order of Planck's time.

4. BLACK-HOLE SINGULARITIES

Since black-hole interiors may be mapped into cosmological models, we may apply the above proof also to black-hole singularities. The easiest example one may work out is the Schwarzschild solution, whose metric near the singularity reads

$$ds^2 = \frac{2m}{r}\,dt^2 - \frac{r}{2m}\,dr^2 + r^2\,d\Omega^2 \tag{44}$$

Inside the horizon, space and time coordinates interchange their roles, making natural the redefinitions of time $T = r^{3/2}(2/9M)^2$ and space $x = t$. In the new coordinates the above line element reads

$$ds^2 = -dT^2 + \left[\frac{4m}{3}\right]^{2/3} T^{-2/3}\,dx^2 + \left[\frac{9m}{2}\right]^{2/3} T^{4/3}\,d\Omega^2 \tag{45}$$

which is a particular case of a Kantowski–Sachs cosmological model (Kantowski and Sachs, 1966; Vajk and Elgroth, 1966). This metric is a solution of Einstein's vacuum equations and, according to the previous discussion, corresponds to $n = 1$. We wish to apply the above ideas to this spacetime. For this end, we perturb this metric, introducing some "test matter" in the form of radiation. Preserving the Kantowski–Sachs form demands $\delta\alpha_2 = \delta\alpha_3$ and also that $n \to 1 + \varepsilon$, with $\varepsilon \ll 1$. Since radiation corresponds to a traceless energy-momentum tensor, $\boldsymbol{\alpha}^2 = n'(2 - n')$ and

$$(\boldsymbol{\alpha} + \delta\boldsymbol{\alpha})^2 = (\alpha_1 + \delta\alpha_1)^2 + 2(\alpha_2 + \delta\alpha_2)^2 = 1 - \varepsilon^2$$

$$(\alpha_1 + \delta\alpha_1) + 2(\alpha_2 + \delta\alpha_2) = 1 + \varepsilon$$

After substituting $\alpha_1 = -2/3$ and $\alpha_2 = 4/3$, we obtain up to the first order in ε

$$\begin{aligned} \delta\alpha_1 &= 2\varepsilon \\ \delta\alpha_2 &= -\varepsilon/2 \end{aligned} \tag{46}$$

Therefore, the perturbed metric reads

$$ds^2 = -dT^2 + \left[\frac{4m}{3}\right]^{2/3} T^{-(2-6\varepsilon)/3}\,dx^2 + \left[\frac{9m}{2}\right]^{2/3} T^{(8-3\varepsilon)/6}\,d\Omega^2 \tag{47}$$

The main ingredients for the evaluation of $f(t_0)$ are the energy and entropy densities

$$\rho = \frac{\varepsilon}{4\pi T^2} \tag{48}$$

$$s = \frac{4\sigma^{1/4}}{3(4\pi)^{3/4}}\,\frac{\varepsilon^{3/4}}{T^{3/2}} \tag{49}$$

and the particle horizon proper distance

$$R = \frac{3T}{4-3\varepsilon} \tag{50}$$

which allow us to evaluate $f(t)$,

$$f(t) \approx \frac{1}{2\pi\varepsilon^{1/4}} \left(\frac{T_p}{T}\right)^{1/2} \tag{51}$$

or, equivalently, in terms of the original coordinate r,

$$F(r) \approx \frac{9}{4\pi\varepsilon^{1/4}} \frac{M_p}{M} \left[\frac{L_p}{r}\right]^{3/4} \tag{52}$$

showing that violation of the EB necessarily occurs for an observer sufficiently close to the singularity.

5. THE THREE-CURVATURE FINITENESS ASSUMPTION

One of the main assumption in the previous sections was that the three-curvature is always well behaved near the singularity. Is this always true and, if not, how does it change our results?

In order to delve into this question, we shall compare the three-geometry Ricci tensor,

$$\begin{aligned} {}^3R_l^l &= -\frac{\lambda^2 a^4 - (\mu b^2 - \nu c^2)^2}{2(abc)^2} \\ {}^3R_m^m &= -\frac{\mu^2 b^4 - (\lambda a^2 - \nu c^2)^2}{2(abc)^2} \\ {}^3R_n^n &= -\frac{\nu^2 c^4 - (\lambda a^2 - \mu b^2)^2}{2(abc)^2} \end{aligned} \tag{53}$$

with the matter energy density, which is assumed to behave ultrarelativistically near the singularity,

$$p \approx \frac{\rho}{3} \tag{54}$$

Under the hypothesis that $T^{0i} \approx 0$, the divergence of the energy-momentum tensor reads

$$T^{0\mu}_{;\mu} \approx T^{00}_{;0} + \Gamma^0_{\mu\nu} T^{\nu\mu} + \Gamma^\nu_{0\nu} T^{00} \tag{55}$$

The Christoffel symbols in a synchronous frame are

$$\Gamma^0_{ij} = \chi_{ij}, \qquad \Gamma^i_{0j} = \chi^i_j, \qquad \Gamma^i_{i0} = \chi \tag{56}$$

so that the above divergence reads

$$\frac{d\rho}{dt}+\frac{2\rho}{3}\chi\approx 0 \tag{57}$$

Due to equation (13)—a consequence of the strong energy condition—we conclude that

$$\rho\approx t^{-2n/3} \tag{58}$$

Comparison of (53) with (58) shows that two possibilities may occur:

1. The three-curvature is either zero or small as compared with the other terms. This happens whenever (a) the parameters λ, μ, ω vanish, corresponding to flat models (Bianchi I); (b) the parameters λ, μ, ν do not vanish and the three α's are positive and lie inside *the stability triangle* (ST), defined as the intersection of (15) with the three planes $\alpha_i=(3n-1)/6$ (see Figure 4). In this situation, the three-curvature behaves as

$$^3R\sim\frac{a^4+b^4+c^4}{(abc)^2}\sim\frac{t^{4\alpha_1}+t^{4\alpha_2}+t^{4\alpha_3}}{t^{2n}}\lll t^{-2n/3}\sim\text{matter}$$

and may be neglected.

2. The point which represents the solution lies outside the stability triangle. In this situation, matter backreaction to the geometry is irrelevant.

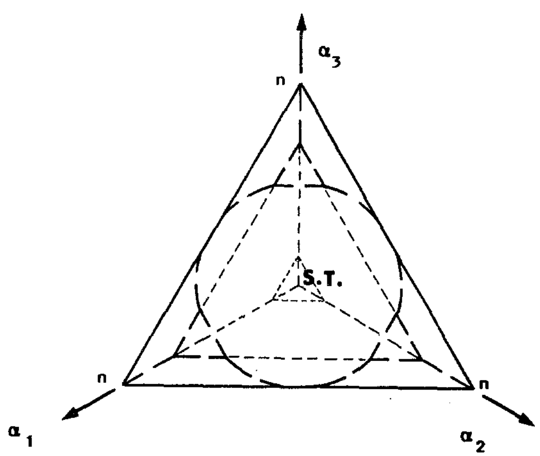

Fig. 4

This is the standard chaotic behavior (Misner, 1969; Belinsky and Khalatnikov, 1969). In this scenario (Landau and Lifshitz, 1970), the gravitational field undergoes transitions between epochs of Kasnerian behavior where the contracting and expanding axes interchange their roles. For any of these epochs the α's may be parametrized as follows:

$$\alpha_1 = \frac{-u}{1+u+u^2}, \qquad \alpha_2 = \frac{1+u}{1+u+u^2}, \qquad \alpha_3 = \frac{u(1+u)}{1+u+u^2} \tag{59}$$

As we move toward the singularity, a given Kasner epoch is determined by the preceding one through the relation $u_{n+1} = u_n - 1$. Moreover, the number of such transitions becomes infinite as the singularity is approached. Thus, for each epoch, similar conclusions as before would follow if it were not for the following circumstance. After very many transitions, necessarily $0 < u \ll 1$ will occur. Whenever this happens, a new era with $u' = 1/u$ (very large) starts (these functions are invariant under the replacement $u \to 1/u$) giving $\alpha_2 \approx -\alpha_1 \approx 0$ and $\alpha_3 \approx 1$. For such a configuration, the particle horizon coordinate behaves as $x_h \sim |\ln t|$, apparently threatening its very existence (for arbitrarily small times). Such a dangerous situation would spoil all the argument based on the EB, for which the particle horizon existence is pivotal. The resolution of this problem lies in the fact that, whenever the gravitational field enters into this configuration, Einstein's equations will drive away from this regime before the singularity is reached. To see explicitly how this happens, let ξ be a time parameter such that $\xi_0 \gg 1$ corresponds to the instant when the system enters into this regime, $\xi \approx 1$ when it leaves it, and $\xi = 0$ the moment the singularity is reached (Belinsky *et al.*, 1970),

$$\xi = \xi_0 \exp\left[\frac{2a_0}{\xi_0}(\theta - \theta_0)\right] \tag{60}$$

where θ is a time parameter defined through

$$abc\, dt = d\theta \tag{61}$$

Inserting these parameters into Einstein's equations, one obtains for this era

$$ab = \frac{a_0^2 \xi}{\xi_0} \tag{62}$$

Next, we calculate the particle horizon proper distance,

$$R(\xi) = c(\xi) \int_0^\xi d\xi\, c^{-1} \frac{dt}{d\xi} \tag{63}$$

However, from (60)-(62) it follows that

$$c^{-1}\frac{dt}{d\xi}=ab\frac{d\theta}{d\xi}=\frac{1}{2}$$

leading to the *exact* expression

$$R(\xi)=\frac{c(\xi)\xi}{2} \tag{64}$$

In the asymptotic region $\xi \gg 1$, the metric coefficients read (Belinsky *et al.*, 1970)

$$\left.\begin{matrix} a \\ b \end{matrix}\right\} = a_0\sqrt{\frac{\xi}{\xi_0}}\left[1\pm\frac{A}{\sqrt{\xi}}\sin(\xi-\xi_0)\right]$$

$$c=c_0\, e^{-A^2(\xi_0-\xi)} \tag{65}$$

where A is some constant. We shall now adopt the strategy of overestimating the particle horizon by assuming this jeopardizing behavior to persist down to the singularity. Thus,

$$R(\xi)<\frac{\xi}{2}c_0\, e^{-A^2(\xi_0-\xi)} \tag{66}$$

On the other hand, adopting an equation of state $p=\gamma\rho$ and assuming matter to be comoving, energy conservation leads to

$$\rho\approx\frac{1}{(abc)^{\gamma+1}}\approx\left[\frac{\xi_0}{\xi}\right]^{\gamma+1} e^{A^2(\xi_0-\xi)(\gamma+1)} \tag{67}$$

Putting all these pieces together, we estimate the function $f(\xi)$ under the hypothesis that S/E peaks for thermal radiation [see equation (37)]

$$f(\xi)=\frac{\hbar S}{2\pi ER}=\frac{2}{3\pi R}\left(\frac{\sigma}{\rho}\right)^{1/4}>\left(\frac{\xi}{\xi_0}\right)^{(\gamma-3)/4} e^{-A^2(\gamma-3)(\xi_0-\xi)/4} \tag{68}$$

On causality grounds, $\gamma\leq 1$ and $f(\xi)>1$ as the singularity is approached ($\xi\to 0$), making transparent the pivotal role causality plays for the sort of argument we are considering in this paper. Thus, we have succeeded in showing that even in the chaotic scenario, the existence of cosmological singularities is ruled out by the entropy bound.

6. SUMMARY AND CONCLUSIONS

The keynote of this paper was to regard the entropy bound as a physical law which supplements the second law of thermodynamics and to apply it in the cosmological context. Since this law refers to closed systems, we applied it to the spacetime region an observer is causally connected to—his particle horizon interior. This was accomplished without making reference either to any particular spacetime model nor to any microscopic description of the matter that fills it. Summarizing our approach, in analogy with Belinsky *et al.*, we expanded the metric around a timelike singularity and assumed quite general and reasonable conditions concerning the nature of the matter content, namely (i) causality and the strong energy condition hold, and (ii) for a given energy density, the matter entropy is always overestimated by the corresponding radiation in thermal equilibrium. Under these assumptions and the hypothesis that the Ricci tensor for spatial sections is always finite, with the aid of Einstein's equation we succeeded in calculating the matter behavior, in constraining the range of the parameters that govern this asymptotic metric, and in calculating the particle horizon proper radius near the singularity.

In light of these results, we showed that the existence of a cosmological singularity is inconsistent with thermodynamics; the entropy bound is violated as the singularity is approached. This violation was shown to occur at time scales of the order of Planck's time, even under the ugly conditions of the chaotic cosmology scenario. This result bears witness to the intuitive feeling that near spacetime singularities, general relativity should be replaced by a quantum theory of gravity.

The prime object of this investigation was to deal with cosmological singularities. However, the present results seem to be correct for any timelike singularity in a globally hyperbolic spacetime; witness the Schwarzschild solution which has been worked out explicitly in Section 4. Had we considered for this problem a coordinate patch which extends analytically beyond the Schwarzschild event horizon, then, the function $f(r)$ could also be analytically extended to this region. Having in mind the structure of equation (51), and on dimensional grounds, this function would have to be the product of M_p/M by a monotonically increasing function $g(l_p/r)$. Thus, even for a distant observer (in the sense that g is small), violation of the entropy bound would occur for black holes which are much lighter than a Planck mass.

As this manuscript was being written, L. Grishchuc pointed out a nice interpretation for our result. For a gas of photons at temperature T and confined in a box of radius R, $S/E \approx T^{-1}$, while the typical photon wavelength is $\lambda \approx T^{-1}$. Thus, the entropy bound (1) may be intuitively understood as the condition that the quantum wavelength should be smaller than the size of the confining box. Enforcing the EB in the context of cosmology, amounts to requiring that the quantum never extends beyond the particle horizon. (If this were not the case, we could envisage a *gedanken* experiment where the collapse of this quantum wave function is used to convey information between causally disconnected regions.) As the singularity is approached, both the particle horizon and the quantum wavelength shrink, but since the former does it faster, sufficiently close to the singularity any quanta will inevitably fall outside this horizon, corresponding to the violation of the entropy bound.

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